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The critical isotherm of the mixed spin Ising model

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Abstract. Series expansions are used to estimate the critical exponent δ for several two- and three-dimensional mixed spin Ising models. The evidence supports the conjecture of extended spin independence. This suggests that our mixed spin models and the standard 'single spin' models share the same value of δ.

Series expansion studies of mixed spin Ising models have until now been restricted to the high-temperature regime (Schofield and Bowers 1981). In order to study the shape of the critical isotherm in such models, we have derived initial terms in the high-field expansion

$$M(u, \mu) = \tfrac{3}{4} - \sum_{n>0} n g_n(u) \mu^n \tag{1}$$

of the reduced magnetisation M. Here the variables $u = \exp(-2J/kT)$ and $\mu = \exp(-mH/kT)$ relate to the Hamiltonian

$$\mathcal{H} = -2J \sum_{\langle ij \rangle} \sigma_i s_j - mH\left(\sum_i \sigma_i + \sum_j s_j\right), \tag{2}$$

in which the notation is standard. This Hamiltonian applies to loose packed lattices in which one sublattice is inhabited by spin-$\frac{1}{2}$ objects ($\sigma_i = \pm\frac{1}{2}$) and the other by spin-1 objects ($s_j = \pm 1$, 0). For the honeycomb (HC) lattice the first ten ferromagnetic polynomials $g_n(u)$ have been obtained by generalising the 'code method' described by Sykes *et al* (1965). For the plane square (SQ), simple cubic (SC), and body centred cubic (BCC) lattices, the first seven of these polynomials have been derived in the same way. (Symmetry is lost because the magnetic sublattices are not identical. This means that the effort needed to reach a given order in (1) is about twice that for the corresponding 'single spin' models.)

Our analysis is based on the series

$$M_c(\mu) = \tfrac{3}{4} - \sum_{n>0} d_n \mu^n, \tag{3}$$

obtained from (1) by setting u equal to its critical value u_c. For the critical value we have used

$$u_c = 0.219\,8334 \text{ (HC)}, \qquad u_c = 0.3588 \pm 0.0025 \text{ (SQ)},$$
$$u_c = 0.5904 \pm 0.0035 \text{ (SC)}, \qquad u_c = 0.6866 \pm 0.0007 \text{ (BCC)}. \tag{4}$$

The first of these follows from the exact result which can be obtained using the star-triangle transformation (Fisher 1959, Yousif 1983). The remainder are estimates which correspond to those obtained after making a few corrections to the high-temperature series expansion analyses of Schofield and Bowers (1981). In table 1 we present the coefficients d_n which we have obtained for the four lattices. For the HC lattice, the above value of u_c is used. For the other lattices, to save space, the coefficients are given only for the central value of the estimates at (4).

Table 1. Magnetisation isotherm coefficients d_n for two- and three-dimensional lattices.

n	HC	SQ	SC	BCC
1	0.056 848	0.072 655	0.124 075	0.135 813
2	0.016 914	0.023 401	0.056 561	0.063 555
3	0.013 975	0.014 609	0.035 768	0.031 663
4	0.011 267	0.011 002	0.025 594	0.026 156
5	0.008 715	0.008 362	0.019 580	0.019 784
6	0.006 799	0.006 953	0.015 265	0.015 842
7	0.005 677	0.005 841	0.013 085	0.014 955
8	0.004 948			
9	0.004 557			
10	0.004 137			

As in the case with spin-$\frac{1}{2}$ (Gaunt and Sykes 1972) and spin-1 (Fox and Gaunt 1972) Ising ferromagnets, the available coefficients d_n are positive for all lattices in all cases. One expects that the series (3) converges up to the critical point $\mu = 1$ $(H = 0)$. The intention is to estimate the critical exponent δ which satisfies the condition $M \sim (1-\mu)^{1/\delta}$ as $\mu \to 1-$. There are several methods which can be tried. The most successful of these are the Padé approximant (PA) method (Baker 1961) and a method due to Gaunt (1967) which uses the coefficients in the series expansion of the logarithmic derivative of $M_c(\mu)$.

In table 2 we give estimates for δ obtained by evaluating PAs to $(1-\mu)\times(\mathrm{d}/\mathrm{d}\mu)\ln M_c(\mu)$ at $\mu = 1$ using the series of table 1. For the SQ, SC and BCC lattices we have obtained such tables for other values of u_c in the ranges (4). There is no space to give them but they are very similar, for each lattice, to the appropriate part of table 2. From all this evidence we estimate

$$\delta = 15.3 \pm 0.5 \text{ (HC)}, \qquad \delta = 14.9 \pm 0.9 \text{ (SQ)},$$
$$\delta = 5.25 \pm 0.45 \text{ (SC)}, \qquad \delta = 5.2 \pm 0.3 \text{ (BCC)}. \tag{5}$$

Gaunt's (1967) method employs the series for

$$-\mu(\mathrm{d}/\mathrm{d}\mu)\ln M_c(\mu) = \sum_{n>0} c_n \mu^n \tag{6}$$

whose coefficients should approach $1/\delta$ as $n \to \infty$. This approach leads to the most consistent results for the spin-$\frac{1}{2}$ (Gaunt 1967, Gaunt and Sykes 1972) and spin-1 (Fox and Gaunt 1970, 1972) Ising ferromagnets. Here, as in the above cases, the values obtained for the coefficients c_n are always positive. In table 3, we present those values of $1/c_n$ which follow from the series of table 1. These are estimates of δ. Again similar results are found, for the SQ, SC and BCC lattices, when the series of (1) are

Table 2. Estimates for δ provided by evaluating the $[D, N]$ PAs to $(1-\mu)(\mathrm{d}/\mathrm{d}\mu)\ln M_c(\mu)$ series at $\mu = 1$ for two- and three-dimensional lattices.

HC

D \ N	1	2	3	4	5	6	7	8
1	17.154	13.428	14.984	14.818	16.055	15.887	15.934	14.903
2	17.666	15.217	15.826	15.603	15.809	16.172	15.700	
3	14.060	15.600	15.623	15.697	15.647	15.438		
4	14.934	15.622	15.594	15.653	15.759			
5	12.814	15.749	15.649	15.620				
6	16.257	15.342	15.491					
7	16.011	15.529						
8	21.653							

SQ

D \ N	1	2	3	4	5
1	14.730	14.452	14.544	14.764	14.799
2	14.388	14.571	15.814	14.879	
3	11.969	14.839	14.893		
4	14.723	14.885			
5	13.873				

SC

D \ N	1	2	3	4	5
1	5.184	5.213	5.243	5.294	5.303
2	5.212	5.089	5.208	5.302	
3	5.242	5.208	5.170		
4	5.272	5.293			
5	5.303				

BCC

D \ N	1	2	3	4	5
1	5.233	5.295	5.062	5.194	5.065
2	5.293	5.253	5.030	5.064	
3	5.119	4.924	5.374		
4	5.059	5.156			
5	5.477				

replaced by series obtained using different estimates of u_c within the ranges (4). From all this evidence we estimate

$$\delta = 15.4 \pm 0.5 \ (\text{HC}), \qquad \delta = 14.9 \pm 0.9 \ (\text{SQ}),$$
$$\delta = 5.3 \pm 0.4 \ (\text{SC}), \qquad \delta = 5.1 \pm 0.5 \ (\text{BCC}). \tag{7}$$

These are in good agreement with the estimates (5).

Table 3. Estimates of δ as calculated from $1/c_n$ for two- and three-dimensional lattices.

n	HC	SQ	SC	BCC
1	13.193	10.323	6.045	5.522
2	19.666	13.930	5.612	4.944
3	16.270	14.617	5.405	5.598
4	14.857	14.403	5.307	5.086
5	14.998	14.831	5.269	5.064
6	15.685	14.752	5.349	5.037
7	15.929	14.872	5.244	4.631
8	15.879			
9	15.317			
10	15.082			

It seems clear that, as expected, δ depends on dimensionality but not on lattice structure. With this assumed, the above estimates may be combined to give

$$\delta = 15.25 \pm 0.35 \text{ (two dimensions)}, \qquad \delta = 5.20 \pm 0.25 \text{ (three dimensions)}. \tag{8}$$

When one compares these with the results of the above references for the spin-$\frac{1}{2}$ and spin-1 Ising ferromagnets, the evidence for an extended form of spin independence is rather good. This is consistent with the view (Schofield and Bowers 1981) that mixed spin Ising models—which, it should be noted, are capable of a particular form of uniaxial ferrimagnetism—belong to the same universality class as the single spin models although they have markedly different translational symmetry.

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